Journal of Engineering Mathematics, Vol. 3, No. 3, July 1969.
Wolters-Noordhoff Publishing – Groningen
Printed in the Netherlands.

An Asymptotic Theory for the Moderately Large Deflections of Anisotropic Plates

O. E. WIDERA

Institut für Umformtechnik, Universität Stuttgart
(Received October 28, 1968)

SUMMARY
In this paper a first approximation theory for the moderately large deflections of anisotropic plates is derived. The method used is that of asymptotic integration of a non-linear set of elasticity equations. Higher order systems can be derived in a systematic manner. For an isotropic material, the derived equations reduce to the von Karman equations.

1. Introduction

Much of the recent activity in the theory of plates and shells has concerned itself with a systematic and consistent derivation of two-dimensional plate and shell equations from the three-dimensional equations of elasticity. The dimensions are introduced via changes in the independent variables and the stress and displacements are expanded in terms of a small geometric parameter. By equating equal powers of the parameter, successive systems of field equations of various orders are obtained. Subsequent integration over the thickness and application of the boundary conditions yields the desired plate or shell approximations. With regard to static plate theory, this method has been used by Goodier [1], Friedrichs and Dressler [2], Reissner [3], Goldenveizer and Kolos [4], Ebcioglu and Habip [5], and Pickett and Johnson [6], among others.

In this paper the method of asymptotic integration is used in order to derive the first approximation equations for a homogeneous anisotropic plate which is described by a partially non-linear set of elasticity equations. The equations obtained contain six elastic constants and are valid for moderately large deflections [7]. For an isotropic material, they reduce to the von Karman equations. For small deflections, the equations uncouple into two equations for the in-plane displacements and one for the transverse displacement.

2. Fundamental Equations

We consider a homogeneous anisotropic plate bounded by a cylindrical surface and by two parallel surfaces perpendicular to the generators of the cylindrical boundary surface. The distance between the two parallel faces is taken as $2h$ and is assumed small compared to a representative distance L along the cylindrical surface. The plate is referred a Cartesian coordinate system x_i $(i=1, 2, 3)$ such that $x_3=0$ represents the middle surface of the plate. Within the framework of a partially non-linear theory of elasticity, the field equations for such a plate are taken as

Strain-displacement equations

$$\begin{aligned} \varepsilon_{\alpha,\beta} &= \tfrac{1}{2}(u_{\alpha,\beta}+u_{\beta,\alpha}+u_{3,\alpha}u_{3,\beta}) \\ \varepsilon_{\alpha 3} &= \tfrac{1}{2}(u_{\alpha,3}+u_{3,\alpha}) \\ \varepsilon_{33} &= u_{3,3} \end{aligned} \tag{1}$$

Equilibrium equations

$$\begin{aligned}&\sigma_{\alpha\beta,\beta}+\sigma_{\alpha 3,3}=0\\&\sigma_{\alpha 3,\alpha}+(\sigma_{\alpha\beta}u_{3,\beta})_{,\alpha}+\sigma_{33,3}=0\end{aligned}\tag{2}$$

Constitutive equations

$$\varepsilon_{ij}=k_{ijkl}\sigma_{kl}\tag{3}$$

Here, σ_{ij} is the symmetric stress tensor, ε_{ij} is the symmetric strain tensor, u_i is the displacement vector, and k_{ijkl} is the elasticity tensor. Latin indices range over the values 1, 2, 3 while Greek indices range from 1 to 2. Repeated indices imply the use of the summation convention. A comma denotes partial differentiation with respect to the indicated coordinate.

It is sometimes convenient to express relations (3) in the contracted from

$$\varepsilon_r=k_{rs}\sigma_s\tag{4}$$

where $r, s=1, 2, 3, 4, 5$ or 6 and the relations between contracted and tensor components are

$$\begin{aligned}&\sigma_1=\sigma_{11},\ \sigma_2=\sigma_{22},\ \sigma_3=\sigma_{33},\ \sigma_4=\sigma_{23},\ \sigma_5=\sigma_{13},\ \sigma_6=\sigma_{12}\,,\\&\varepsilon_1=\varepsilon_{11},\ \varepsilon_2=\varepsilon_{22},\ \varepsilon_3=\varepsilon_{33},\ \varepsilon_4=2\varepsilon_{23},\ \varepsilon_5=2\varepsilon_{13},\ \varepsilon_6=2\varepsilon_{12}\,,\end{aligned}\tag{5}$$

For simplicity, the surface $x_3=\pm h$ are assumed to be free from surface traction,

$$\sigma_{13}=\sigma_{23}=\sigma_{33}=0\,.\tag{6}$$

3. Asymptotic Approximation

We introduce dimensionless coordinates ξ_i defined by

$$x_\alpha=L\xi_\alpha\,,\quad x_3=h\xi_3\,.\tag{7}$$

We introduce a geometric parameter λ,

$$\lambda=\frac{h}{L}\ll 1\,.\tag{8}$$

Dimensionless stresses s_{ij} are defined by

$$\sigma_{\alpha\beta}=\sigma s_{\alpha\beta},\quad \sigma_{\alpha 3}=\lambda\sigma s_{\alpha 3},\quad \sigma_{33}=\lambda^2\sigma s_{33}\tag{9}$$

where σ is a representative stress level.

The elasticity constants k_{ijkl} will in general not all be of the same order. It is therefore assumed that these functions can be written as a finite sum

$$k_{ijkl}=K\sum_{n=0}^{N}k_{ijkl}^{(n)}\lambda^n\tag{10}$$

where K is a scale factor and $k_{ijkl}^{(n)}=0(1)$ or vanish identically.

Dimensionless displacements v_i are now defined as

$$u_\alpha=K\sigma Lv_\alpha,\quad u_3=K\sigma L\lambda^{-1}v_3\,.\tag{11}$$

In terms of these dimensionless variables, the fundamental equations can be rewritten as follows:

$$\begin{aligned}v_{3,3}&=\lambda^2\sum_{n=0}^{N}\lambda^n\left[k_{31}^{(n)}\ s_{11}+k_{32}^{(n)}s_{22}+k_{33}^{(n)}s_{33}+k_{34}^{(n)}s_{23}+k_{35}^{(n)}s_{13}+k_{36}^{(n)}s_{12}\right]\\v_{1,3}&=-v_{3,1}+\lambda\Sigma\lambda^n\left[k_{51}^{(n)}s_{11}+\ldots+k_{56}^{(n)}s_{12}\right]\\v_{2,3}&=-v_{3,2}+\lambda\Sigma\lambda^n\left[k_{41}^{(n)}s_{11}+\ldots+k_{46}^{(n)}s_{12}\right]\end{aligned}$$

$$v_{1,1} = -\frac{\gamma}{2\lambda^2} v_{3,1}^2 + \Sigma \lambda^n [k_{11}^{(n)} s_{11} + \ldots + k_{16}^{(n)} s_{12}]$$

$$v_{2,2} = -\frac{\gamma}{2\lambda^2} v_{3,2}^2 + \Sigma \lambda^n [k_{21}^{(n)} s_{11} + \ldots + k_{26}^{(n)} s_{12}]$$

$$v_{1,2} + v_{2,1} = -\frac{\gamma}{\lambda^2} v_{3,1} v_{3,2} + \Sigma \lambda^n [k_{61}^{(n)} s_{11} + \ldots + k_{66}^{(n)} s_{12}]$$

$$s_{13,3} = -s_{11,1} - s_{12,2}$$

$$s_{23,3} = -s_{12,1} - s_{22,2}$$

$$s_{33,3} = -s_{13,1} - s_{23,2} - \frac{\gamma}{\lambda^2} [(s_{11} v_{3,1} + s_{12} v_{3,2})_{,1} + (s_{12} v_{3,1} + s_{22} v_{3,2})_{,2}] \tag{12}$$

where the contracted form (4) has been used and

$$\gamma = K\sigma\,. \tag{13}$$

Assuming it to be possible, we expand, in view of condition (8), the stresses and displacements in terms of a power series in λ^2,

$$s_{ij} = \sum_{m=1}^{M} s_{ij}^{(m-1)} \lambda^{2m} \tag{14}$$

$$v_i = \sum_{m=1}^{M} v_i^{(m-1)} \lambda^{2m}$$

where $s_{ij}^{(m-1)}(\xi_k)$, $v_i^{(m-1)}(\xi_k)$ do not depend on λ for $m=1, 2, \ldots, M-1$ and the remainders $s_{ij}^{(M)}(\xi_k, \lambda)$, $v_i^{(M)}(\xi_k, \lambda)$ are assumed to be such that they tend to a finite limit as λ approaches zero.

If we now substitute expansions (14) into equations (12) we obtain sequences of systems of equations which are integrable with respect to ξ_3 in a step by step manner and thus represent the desired two-dimensional plate approximations. The first system of equations represents the simplest thin plate equations. The higher order systems yield the higher order terms in expansions (14) and represent thickness effects. In the following, only the first approximation will be considered. These equations are

$$v_{3,3} = 0 \tag{15}$$

$$v_{1,3} = -v_{3,1} \tag{16}$$

$$v_{2,3} = -v_{3,2} \tag{17}$$

$$v_{1,1} = -\frac{\gamma}{2} v_{3,1}^2 + k_{11} s_{11} + k_{12} s_{22} + k_{16} s_{12} \tag{18}$$

$$v_{2,2} = -\frac{\gamma}{2} v_{3,2}^2 + k_{21} s_{11} + k_{22} s_{22} + k_{26} s_{12} \tag{19}$$

$$v_{1,2} + v_{2,1} = -\gamma v_{3,1} v_{3,2} + k_{61} s_{11} + k_{62} s_{22} + k_{66} s_{12} \tag{20}$$

$$s_{13,3} = -s_{11,1} - s_{12,2} \tag{21}$$

$$s_{23,3} = -s_{12,1} - s_{22,2} \tag{22}$$

$$s_{33,3} = -s_{13,1} - s_{23,2} - \gamma [(s_{11} v_{3,1} + s_{12} v_{3,2})_{,1} + (s_{12} v_{3,1} + s_{22} v_{3,2})_{,2}]\,. \tag{23}$$

With respect to equations (15–23), it is to be noted that only six elastic constants enter into the first approximation. The effects of the other moduli are of higher order in λ and enter into the systems of higher order. It is further to be noted that the asymptotic method used here leads to a solution giving all stress coefficients, including those of transverse shear and normal stress.

The corresponding boundary conditions for (15–23) are

$$s_{13} = s_{23} = s_{33} = 0 \qquad (\xi_3 = \pm 1) \tag{24}$$

4. Thin Plate Equations

Integration of (15–17) with respect to ξ_3 yields

$$\begin{aligned} v_3 &= V_3(\xi_1, \xi_2) \\ v_1 &= V_1(\xi_1, \xi_2) - V_{3,1}\xi_3 \\ v_2 &= V_2(\xi_1, \xi_2) - V_{3,2}\xi_3 , \end{aligned} \tag{25}$$

where V_1, V_2, V_3 are the middle surface displacement components of the plate.

Let us define a matrix $[k_1]$ as follows:

$$[k_1] = \begin{bmatrix} k_{11} & k_{12} & k_{16} \\ k_{21} & k_{22} & k_{26} \\ k_{61} & k_{62} & k_{66} \end{bmatrix} \tag{26}$$

and let $[E]$ be the inverse of $[k_1]$

$$[E] = [k_1]^{-1} = \begin{bmatrix} E_{11} & E_{12} & E_{13} \\ E_{21} & E_{22} & E_{32} \\ E_{31} & E_{32} & E_{33} \end{bmatrix} \tag{27}$$

Equations (18–20) then yield the following expressions for the stresses:

$$\begin{bmatrix} s_{11} \\ s_{22} \\ s_{12} \end{bmatrix} = [E] \begin{bmatrix} \varepsilon_1 + K_1\xi_3 \\ \varepsilon_2 + K_2\xi_3 \\ \varepsilon_{12} + K_{12}\xi_3 \end{bmatrix} \tag{28}$$

where

$$\begin{aligned} \varepsilon_1 &= V_{1,1} + \frac{\gamma}{2} V_{3,1}^2 \\ \varepsilon_2 &= V_{2,2} + \frac{\gamma}{2} V_{3,2}^2 \\ \varepsilon_{12} &= V_{1,2} + V_{2,1} + \gamma V_{3,1} V_{3,2} \end{aligned} \tag{29}$$

and

$$\begin{aligned} K_1 &= -V_{3,11} \\ K_2 &= -V_{3,22} \\ K_{12} &= -2V_{3,12} \end{aligned} \tag{30}$$

As defined by (29) and (30), the ε's represent the strains of the middle surface while the K's represent the curvatures.

Integration of (21–23) yields

$$\begin{aligned} s_{13} = S_{13}(\xi_1, \xi_2) &- [E_{11}\varepsilon_{1,1} + E_{12}\varepsilon_{2,1} + E_{13}\varepsilon_{12,1} + E_{31}\varepsilon_{1,2} + E_{32}\varepsilon_{2,2} + E_{33}\varepsilon_{12,2}]\xi_3 \\ &- [E_{11}K_{1,1} + E_{12}K_{2,1} + E_{12}K_{12,1} + E_{31}K_{1,2} + E_{32}K_{2,2} + E_{33}K_{12,2}]\frac{\xi_3^2}{2} \end{aligned} \tag{31}$$

$$s_{23} = S_{23}(\xi_1, \xi_2) - [E_{31}\varepsilon_{1,1} + E_{32}\varepsilon_{2,1} + E_{33}\varepsilon_{12,1} + E_{21}\varepsilon_{1,2} + E_{22}\varepsilon_{2,2} + E_{23}\varepsilon_{12,2}]\xi_3 - [E_{31}K_{1,1} + E_{32}K_{2,1} + E_{33}K_{12,1} + E_{21}K_{1,2} + E_{22}K_{2,2} + E_{23}K_{12,2}]\frac{\xi_3^2}{2} \tag{32}$$

$$\begin{aligned} s_{33} = {} & S_{33}(\xi_1, \xi_2) - [S_{13,1} + S_{23,2}]\xi_3 + [E_{11}\varepsilon_{1,11} + E_{12}\varepsilon_{2,11} + E_{13}\varepsilon_{12,11} + 2E_{31}\varepsilon_{1,12} \\ & + 2E_{32}\varepsilon_{2,12} + 2E_{33}\varepsilon_{12,12} + E_{21}\varepsilon_{1,22} + E_{22}\varepsilon_{2,22} + E_{23}\varepsilon_{12,22}]\frac{\xi_3^2}{2} + \\ & + [E_{11}K_{1,11} + E_{12}K_{2,11} + E_{13}K_{12,11} + 2E_{31}K_{1,12} + 2E_{33}K_{12,12} + 2E_{32}K_{2,12} + \\ & + E_{21}K_{1,22} + E_{22}K_{2,22} + E_{23}K_{12,22}]\frac{\xi_3^3}{6} - \\ & - \gamma\{[(E_{11}\varepsilon_1 + E_{12}\varepsilon_2 + E_{13}\varepsilon_{12})V_{3,1} + (E_{31}\varepsilon_1 + E_{32}\varepsilon_2 + E_{33}\varepsilon_{12})V_{3,2}]_{,1} + \\ & + [(E_{31}\varepsilon_1 + E_{32}\varepsilon_2 + E_{33}\varepsilon_{12})V_{3,1} + (E_{21}\varepsilon_1 + E_{22}\varepsilon_2 + E_{23}\varepsilon_{12})V_{3,2}]_{,2}\}\xi_3 - \\ & - \gamma\{[(E_{11}K_1 + E_{12}K_2 + E_{13}K_{12})V_{3,1} + (E_{31}K_1 + E_{32}K_2 + E_{33}K_{12})V_{3,2}]_{,1} + \\ & + [(E_{31}K_1 + E_{32}K_2 + E_{33}K_{12})V_{3,1} + (E_{21}K_1 + E_{22}K_2 + E_{23}K_{12})V_{3,2}]_{,2}\}\frac{\xi_3^2}{2} \end{aligned} \tag{33}$$

where S_{13}, S_{23}, S_{33} are the middle surface stresses.

Satisfaction of boundary conditions (24) yields

$$S_{13} = \tfrac{1}{2}(E_{11}K_{1,1} + E_{12}K_{2,1} + E_{13}K_{12,1} + E_{31}K_{1,2} + E_{32}K_{2,2} + E_{33}K_{12,2}) \tag{34}$$

$$S_{23} = \tfrac{1}{2}(E_{31}K_{1,1} + E_{32}K_{2,2} + E_{33}K_{12,2} + E_{21}K_{1,2} + E_{22}K_{2,2} + E_{23}K_{12,2}) \tag{35}$$

$$\begin{aligned} S_{33} = {} & -\tfrac{1}{2}(E_{11}\varepsilon_{1,11} + E_{12}\varepsilon_{2,11} + E_{13}\varepsilon_{12,11} + 2E_{31}\varepsilon_{1,12} + 2E_{32}\varepsilon_{2,12} + 2E_{33}\varepsilon_{12,12} + \\ & + E_{21}\varepsilon_{1,22} + E_{22}\varepsilon_{2,22} + E_{23}\varepsilon_{12,22}) + \frac{\gamma}{2}\{[(E_{11}K_1 + E_{12}K_2 + E_{13}K_{12}]V_{3,1} + \\ & + (E_{31}K_1 + E_{32}K_2 + E_{33}K_{12})V_{3,2}]_{,1} + [(E_{31}K_1 + E_{32}K_2 + E_{33}K_{12})V_{3,1} + \\ & + (E_{21}K_1 + E_{22}K_2 + E_{23}K_{12})V_{3,2}]_{,2}\} \end{aligned} \tag{36}$$

and the following equations for the middle surface displacements:

$$\begin{aligned} & E_{11}V_{1,11} + (E_{13} + E_{31})V_{1,12} + E_{33}V_{1,22} + E_{13}V_{2,11} + (E_{12} + E_{33})V_{2,12} + E_{32}V_{2,22} \\ & = -\gamma\{(E_{11}V_{3,1} + E_{13}V_{3,2})V_{3,11} + [(E_{12} + E_{33})V_{3,2} + (E_{13} + E_{31})V_{3,1}]V_{3,12} + \\ & + (E_{33}V_{3,1} + E_{32}V_{3,2})V_{3,22}\} \end{aligned} \tag{37}$$

$$\begin{aligned} & E_{31}V_{1,11} + (E_{21} + E_{33})V_{1,12} + E_{23}V_{1,22} + E_{33}V_{2,11} + (E_{23} + E_{32})V_{2,12} + E_{22}V_{2,22} \\ & = -\gamma\{(E_{31}V_{3,1} + E_{33}V_{3,2})V_{3,11} + [(E_{23} + E_{32})V_{3,2} + (E_{21} + E_{33})V_{3,1}]V_{3,12} + \\ & + (E_{23}V_{3,1} + E_{22}V_{3,2})V_{3,22}\} \end{aligned} \tag{38}$$

$$\begin{aligned} & E_{11}V_{3,1111} + 2(E_{13} + E_{31})V_{3,1112} + (4E_{33} + E_{12} + E_{21})V_{3,1122} \\ & + 2(E_{23} + E_{32})V_{3,1222} + E_{22}V_{3,2222} \\ & = 3\gamma\{[(E_{11}\varepsilon_1 + E_{12}\varepsilon_2 + E_{13}\varepsilon_{12})V_{3,1} + (E_{31}\varepsilon_1 + \varepsilon_{32}\varepsilon_2 + E_{33}\varepsilon_{12})V_{3,2}]_{,1} \\ & + [(E_{31}\varepsilon_1 + E_{32}\varepsilon_2 + E_{33}\varepsilon_{12})V_{3,1} + (E_{21}\varepsilon_1 + E_{22}\varepsilon_2 + E_{23}\varepsilon_{12})V_{3,2}]_{,2}\}. \end{aligned} \tag{39}$$

Equations (25), (28), (31–39) comprise a complete set of equations for the determination of the first approximation stresses and displacements.

For an isotropic material, matrix $[E]$ is given by

$$[E] = \begin{bmatrix} \frac{EK}{1-\nu^2} & \frac{\nu EK}{1-\nu^2} & 0 \\ \frac{\nu EK}{1-\nu^2} & \frac{EK}{1-\nu^2} & 0 \\ 0 & 0 & \frac{EK}{2(1+\nu)} \end{bmatrix}$$

In this case, equations (37–39) reduce to the well-known von Karman equations of thin plate theory. In the above, ν is Poisson's ratio and E is the modulus of elasticity.

5. Conclusion

In this paper a first approximation theory for the moderately large deflections of a homogeneous anisotropic plate was derived by use of the method of asymptotic integration of a partially non-linear set of elasticity equations. Even though complete anisotropy was allowed for in the elasticity equations, the first approximation equations contain only six elastic constants. This implies that, to a first approximation, the plate material has a plane of elastic symmetry parallel to the middle surface.

Although not considered here, the higher order approximations will introduce the effect of transverse shear and normal stress in the deformation of the plate in a systematic and consistent manner. This was shown in a previous report on shell dynamics [8]. The limiting case of small deflections can be obtained by setting the parameter γ equal to zero.

The support of the Alexander von Humboldt Stiftung is gratefully acknowledged.

REFERENCES

[1] J. N. Goodier, On the problems of the beam and the plate in the theory of elasticity, *Trans. Roy. Soc. Canada*, 32 (1938) 65–88.

[2] K. O. Friedrichs and R. F. Dressler, A boundary layer theory for elastic plates, *Com. Pure Appl. Math.*, 14 (1961) 1–33.

[3] E. Reissner, On the derivation of boundary conditions for plate theory, *Proc. Roy. Soc. London*, A276 (1963) 178–186.

[4] A. L. Goldenveizer, and A. V. Kolos, On the derivation of two-dimensional equations in the theory of thin elastic plates, *PMM*, 29 (1965) 141–155.

[5] I. K. Ebcioglu, and L. M. Habip, Moderately large deflection of a transversely isotropic plate, *Proc. 9th Midwestern Mech. Conf.*, Madison, Wisc., 1965.

[6] G. Pickett and M. W. Johnson, *Analytical procedures for predicting the mechanical properties of fiber reinforced composites*, Tech. Rept. AFML-TR-65-220, Wright-Patterson A. F. base, Ohio, 1966.

[7] V. V. Novozhilov, *Foundations of the nonlinear theory of elasticity*, Graylock Press, Rochester, N.Y., 1953.

[8] M. W. Johnson and O. E. Widera, An Asymptotic Dynamic Theory for Cylindrical Shells, *Studies in Appl. Math.*, to be published.